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The Knaster–Kuratowski–Mazurkiewicz theorem and abstract convexities

George L. Cain Jr. [a], Luis González [b,*]

[a] *Department of Mathematics, Georgia Institute of Technology, Atlanta, GA, USA*
[b] *Departamento de Matematicas, Facultad de Ciencias, Universidad de los Andes, Mérida, Mérida, Venezuela*

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Abstract

The Knaster–Kuratowski–Mazurkiewicz covering theorem (KKM), is the basic ingredient in the proofs of many so-called "intersection" theorems and related fixed point theorems (including the famous Brouwer fixed point theorem). The KKM theorem was extended from R^n to Hausdorff linear spaces by Ky Fan. There has subsequently been a plethora of attempts at extending the KKM type results to arbitrary topological spaces. Virtually all these involve the introduction of some sort of abstract convexity structure for a topological space, among others we could mention H-spaces and G-spaces. We have introduced a new abstract convexity structure that generalizes the concept of a metric space with a convex structure, introduced by E. Michael in [E. Michael, Convex structures and continuous selections, Canad. J. Math. 11 (1959) 556–575] and called a topological space endowed with this structure an M-space. In an article by Shie Park and Hoonjoo Kim [S. Park, H. Kim, Coincidence theorems for admissible multifunctions on generalized convex spaces, J. Math. Anal. Appl. 197 (1996) 173–187], the concepts of G-spaces and metric spaces with Michael's convex structure, were mentioned together but no kind of relationship was shown. In this article, we prove that G-spaces and M-spaces are close related. We also introduce here the concept of an L-space, which is inspired in the MC-spaces of J.V. Llinares [J.V. Llinares, Unified treatment of the problem of existence of maximal elements in binary relations: A characterization, J. Math. Econom. 29 (1998) 285–302], and establish relationships between the convexities of these spaces with the spaces previously mentioned.

Keywords: KKM; Abstract convexities

1. The Knaster–Kuratowski–Mazurkiewicz theorem and some generalizations

In this section we first review earlier results in the area of the so-called intersection theorems, including the original Knaster–Kuratowski–Mazurkiewicz theorem (KKM) [3] and a further developments by K. Fan. We then review and give a somewhat improved version of a KKM type result of C. Horvath [2].

* Corresponding author.
E-mail addresses: cain@math.gatech.edu (G.L. Cain Jr.), lgonzal@ula.ve (L. González).

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Notation. The convex hull of a subset A of a linear space will be denoted by $[A]$, and given a set X, $\langle X\rangle$ will denote the family of all finite sets of X.

Theorem 1.1 *(KKM). (See [3].) Let $P_n = [\{a_1,\dots,a_{n+1}\}]$ be a closed n-simplex and let $F_1,\dots,F_{n+1}$, be $n+1$ closed subsets of P_n. If for each set $\{i,j,\dots,l\}\subset\{1,\dots,n+1\}$ we have*

$$\bigl[\{a_i,a_j,\dots,a_l\}\bigr]\subset F_i\cup F_j\cup\cdots\cup F_l, \tag{1}$$

then $\bigcap\{F_i\colon i=1,\dots,n+1\}\neq\emptyset$.

Theorem 1.2. *Let X be an arbitrary set in a topological vector space Y, and let $F : X\to 2^Y$ be a closed valued multifunction such that*

(1) *for any set $J\in\langle X\rangle$ we have that $[J]\subset\bigcup\{F(x)\colon x\in J\}$;*
(2) *$F(x)$ is compact for at least one $x\in X$,*

then $\bigcap\{F(x)\colon x\in X\}\neq\emptyset$.

Notation. The closed n-simplex of R^{n+1} whose vertices are the set $\{e_1,e_2,\dots,e_{n+1}\}$ where $e_i=(0,\dots,1,\dots,0)$ with the one in the ith component will be denoted by Δ_n, that is, $\Delta_n=[\{e_1,\dots,e_{n+1}\}]$.

The following two theorems are due to C. Horvath [2].

Theorem 1.3. *Suppose X is a topological space and $Z_{n+1}=\{i\in Z\colon 1\leqslant i\leqslant n+1\}$. If $\Gamma:\langle Z_{n+1}\rangle\to X$ is a multifunction such that*

(1) *each $\Gamma(J)$ is contractible*; *and*
(2) *$\Gamma(J)\subset\Gamma(K)$ whenever $J\subset K$,*

then there is a continuous function $f:\Delta_n\to X$ such that $f([\{e_i\colon i\in J\}])\subset\Gamma(J)$.

Proof. Let $S_0\subset\Delta_n$ be given by

$$S_0=\bigcup\bigl\{\bigl[\{e_i\}\bigr]\colon i\in Z_{n+1}\bigr\}$$

and define $f_0:S_0\to X$ simply by setting $f(e_i)=x_i\in\Gamma(\{i\})$. Then f_0 is clearly continuous. Now let $S_1\subset\Delta_n$ be given by

$$S_1=\bigcup\bigl\{\bigl[\{e_i,e_j\}\bigr]\colon i,j\in Z_{n+1};\ i\neq j\bigr\}.$$

We know that $f([e_i])\in\Gamma(\{i\})\subset\Gamma(\{i,j\})$ and $f([e_j])\in\Gamma(\{j\})\subset\Gamma(\{i,j\})$, therefore there is a continuous extension of f_0, say $f_1^{\{i,j\}}:[\{e_i,e_j\}]\to\Gamma(\{i,j\})$, for all $i,j\in Z_{n+1}$; $i\neq j$. Define $f_1:S_1\to X$ by $f_1|[\{e_i,e_j\}]=f_1^{\{i,j\}}$. Then f_1 is well-defined continuous; and $f_1([e_i])\in\Gamma(\{i\})$, for all $i\in Z_{n+1}$, and $f_1([\{e_i,e_j\}])\subset\Gamma(\{i,j\})$, for all $i,j\in Z_{n+1}; i\neq j$.

Let $k<n+1$ and $S_k\subset\Delta_n$ be given by

$$S_k=\bigcup\bigl\{\bigl[\{e_i\colon i\in J\}\bigr]\colon J\subset Z_{n+1},\ |J|\leqslant k\bigr\}\quad\bigl(\text{where } |J| \text{ means cardinality of } J\bigr).$$

Suppose there is a continuous $f_k:S_k\to X$ such that $f_k([K])\subset\Gamma(K)$ for all $K\subset J$. Let

$$S_{k+1}=\bigcup\bigl\{\bigl[\{e_i\colon i\in L\}\bigr]\colon L\subset Z_{n+1},\ |L|\leqslant k+1\bigr\}.$$

Suppose $|L|=k+1$. Let $[L']=[\{e_{i_1},\dots,e_{i_{k+1}}\}]\in S_{k+1}$. The boundary of $[L']$ is $B=\bigcup\{[\{e_{i_1},\dots,\hat e_{i_j},\dots,e_{i_{k+1}}\}]\colon j=1,\dots,k+1\}$. Since each $f_k([\{e_{i_1},\dots,\hat e_{i_j},\dots,e_{i_{k+1}}\}])\subset\Gamma(L)$, thus there is a continuous extension of f_k, say $f_{k+1}^L:[L']\to\Gamma(L)$. We now define $f_{k+1}:S_{k+1}\to X$ by $f_{k+1}|[L]=f_{k+1}^L$. This gives a well-defined continuous extension of f_k to S_{k+1}. We can take $f\equiv f_{n+1}$, and the theorem is proved. □

Theorem 1.4. *Suppose $\{F_i\colon\ i = 1, \dots, n+1\}$ is a collection of closed subsets of a topological space X and $\Gamma : \langle Z_{n+1}\rangle \to X$ is a multifunction such that each $\Gamma(J)$ is contractible and $\Gamma(J) \subset \Gamma(K)$ whenever $J \subset K$. Suppose moreover that $\Gamma(J) \subset \bigcup\{F_i\colon\ i \in J\}$ for each $J \subset Z_{n+1}$. Then $\bigcap\{F_i\colon\ i = 1, \dots, n+1\} \neq \emptyset$.*

Proof. From Theorem 1.3, there exists a continuous function $f : \Delta_n \to X$ such that $f([\{e_i\colon\ i \in J\}]) \subset \Gamma(J) \subset \bigcup\{F_i\colon\ i \in J\}$.

Let $G_i = f^{-1}(F_i)$; $i \in Z_{n+1}$, and let $J \subset Z_{n+1}$; $J \neq \emptyset$. By continuity of f, we have that G_i is a closed set of Δ_n for $i \in Z_{n+1}$, and besides we have that

$$\big[\{e_i\colon\ i \in J\}\big] \subset f^{-1}\big(\Gamma(J)\big) \subset f^{-1}\Big(\bigcup\{F_i\colon\ i \in J\}\Big) = \bigcup\big\{f^{-1}(F_i)\colon\ i \in J\big\} = \bigcup\{G_i\colon\ i \in J\}.$$

Then, by Theorem 1.1 we have that $\bigcap\{G_i\colon\ i = 1, \dots, n+1\} \neq \emptyset$. This implies that $\bigcap\{F_i\colon\ i = 1, \dots, n+1\} \neq \emptyset$. □

The following corollary is due to Horvath [2].

Corollary 1.5. *Let Y be a compact space, X an arbitrary set and $F : X \to 2^Y$ a closed valued multifunction. Suppose that $\Gamma : \langle X\rangle \to 2^Y$ is a multifunction such that*

(1) *$\Gamma(A)$ is contractible for all $A \in \langle X\rangle$;*
(2) *for any $A, B \in \langle X\rangle$, $A \subset B$ implies $\Gamma(A) \subset \Gamma(B)$;*
(3) *for any $A \in \langle X\rangle$, $\Gamma(A) \subset F(A)$.*

Then $\bigcap\{F(x)\colon\ x \in X\} \neq \emptyset$.

Proof. We shall show that the collection $\mathbf{F} = \{F(x)\colon\ x \in X\}$ of closed subsets of the compact space Y has the finite intersection property. To this end, let $\{F(x_i)\colon\ i = 1, \dots, n+1\}$ a finite subset of $\mathbf{F}$, and consider the multifunction $\Gamma' : \langle Z_{n+1}\rangle \to X$ defined by $\Gamma'(J) = \Gamma(\{x_i\colon\ i \in J\})$ for $J \in \langle Z_{n+1}\rangle$.

Now if $J, K \in \langle Z_{n+1}\rangle$, $J \subset K$; we have that

$$\Gamma'(J) = \Gamma\big(\{x_i\colon\ i \in J\}\big) \subset \Gamma\big(\{x_i\colon\ i \in K\}\big) = \Gamma'(K).$$

On the other hand, $\Gamma'(J) = \Gamma(\{x_i\colon\ i \in J\}) \subset \bigcup\{F(x_i)\colon\ i \in J\}$. Therefore, the multifunction Γ' and the collection $\{F(x_i)\colon\ i = 1, \dots, n+1\}$ satisfy the hypotheses of Theorem 1.4. Hence $\bigcap\{F(x_i)\colon\ i = 1, \dots, n+1\} \neq \emptyset$.

Thus, $\mathbf{F}$ is a collection of closed sets satisfying the finite intersection property. From the compactness of Y, we conclude that $\bigcap\{F(x)\colon\ x \in X\} \neq \emptyset$. □

2. Spaces with abstracts convexities

The original KKM theorem, and its early extensions, depend on the idea of the convex hull of a finite set. Subsequent extensions of these results to topological spaces without a linear structure have depended on various, more or less, ad hoc “convexity” notions. This work also has led to the definition of several types of abstract convexities. In this section, we review two of these and present two of our own. In addition, we introduce for each of these the idea of convex set.

The concept of H-space is due to Bardaro and Cepitelli in [1].

Definition 2.1. A triple (X, D, Γ), in which X is a topological space, D is a nonempty subset of X, and $\Gamma\langle D\rangle \to 2^X$ is a function from the collection of nonempty finite subsets of D into 2^X such that

(1) $\Gamma(A)$ is contractible for all $A \in \langle D\rangle$;
(2) $\Gamma(A) \subset \Gamma(B)$ whenever $A \subset B$

is called an *H-space*. In case $D = X$ we simply write (X, Γ) for (X, D, Γ).

Definition 2.2. A subset $S \subset X$, where (X, D, Γ) is an H-space, is called *H-convex* if for each $A \in \langle D \cap S\rangle$, it is true that $\Gamma(A) \subset S$.

The concept of G-space was introduced by Park and Kim in [6].

Definition 2.3. A triple (X, D, Γ) in which X is a topological space, D is a nonempty subset of X, and $\Gamma : \langle D\rangle \to 2^X$ is a function from the set $\langle D\rangle$ of nonempty finite subsets of D into 2^X such that

(1) $\Gamma(A) \subset \Gamma(B)$ whenever $A \subset B$;
(2) For each $A = \{a_1, \ldots, a_{n+1}\} \in \langle D\rangle$, and any indexing or labeling of A, there is a continuous function $\phi_A : \overline{\Delta}_n \to \Gamma(A)$ such that for any subset $B = \{a_{i_1}, \ldots, a_{i_m}\} \subset A$, we have

$$\phi_A\big([e_{i_1}, \ldots, e_{i_m}]\big) \subset \Gamma(B),$$

is called a *G-space*. Again, in case $D = X$, we write simply $(X, D, \Gamma) = (X, \Gamma)$.

Definition 2.4. Let (X, D, Γ) be a G-space. A subset S of X is said to be *G-convex* if $A \in \langle D \cap S\rangle$, then $\Gamma(A) \subset S$.

Notice that the definitions of an H-space and a G-space are very similar. The definition of a G-space results from that of an H-space, by replacing the requirement that $\Gamma(A)$ be contractible with the requirement that there exists a function ϕ_A with the given properties. Not surprisingly, the two concepts are intimately related. More on this later.

In the course of his work, E. Michael [5] introduced the notion of a convex structure on a metric space (X, ρ).

Notation. Given any integer $m \geqslant 2$ and $1 \leqslant i \leqslant m$, let $\delta_i : R^{m+1} \to R^m$ denote the function defined by $\delta_i(x_1, \ldots, x_m) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$.

Definition 2.5. A *convex structure* on a metric space (X, ρ) assigns to each positive integer n a subset $M_n \subset X^n$, and a function $k_{n+1} : M_{n+1} \times \overline{\Delta}_n \to X$, such that

(1) If $x \in M_1$, then $k_1(x, 1) = x$.
(2) If $x \in M_{n+1} (n \geqslant 1)$ and $i \leqslant n + 1$, then $\delta_i(x) \in M_n$ and for any $t \in \overline{\Delta}_n$ with $t_i = 0$, $k_{n+1}(x, t) = k_n(\delta_i(x), \delta_i(t))$.
(3) If $x \in M_{n+1} (n \geqslant 1)$ with $x_i = x_{i+1}$ for some $i < n + 1$ and if $t \in \overline{\Delta}_n$, then $k_{n+1}(x, t) = k_n(\delta_i(x), t^*)$, where $t^* = (t_0, \ldots, t_{i-1}, t_i + t_{i+1}, \ldots, t_n)$.
(4) If $x \in M_{n+1}$, then the map $t \to k_{n+1}(x, t)$, from $\overline{\Delta}_n$ to X, is continuous.
(5) For all $\epsilon \geqslant 0$ there exists a neighborhood V_ϵ of the diagonal in $X \times X$ such that, for all n and all $x, y \in M_{n+1}$, $(x_i, y_i) \in V_\epsilon$ for $i = 1, \ldots, n + 1$, implies $\rho(k_{n+1}(x, t), k_{n+1}(y, t)) < \epsilon$ for all $t \in \overline{\Delta}_n$.

Michael's convex structure inspired the following definition.

Definition 2.6. A Michael space or *M-space* is a triple $(X, \mathbf{M}, \mathbf{k})$, where X is a topological space, $\mathbf{M} = \{M_n$: n integer, $n \geqslant 1\}$ is a collection of sets where $M_n \subset X^n$ for all $n \geqslant 1$, and $\mathbf{k} = \{k_n$: n integer, $n \geqslant 1\}$ is a collection of functions satisfying

(1) $k_{n+1} : M_{n+1} \times \overline{\Delta}_n \to X$.
(2) If $x \in M_{n+1} (n \geqslant 1)$ and $i \leqslant n + 1$, then $\delta_i(x) \in M_n$ and for any $t \in \overline{\Delta}_n$ with $t_i = 0$, $k_{n+1}(x, t) = k_n(\delta_i(x), \delta_i(t))$.
(3) If $x \in M_{n+1}$, then the map $t \to k_{n+1}(x, t)$, from $\overline{\Delta}_n$ to X, is continuous.

Definition 2.7. Let $(X, \mathbf{M}, \mathbf{k})$ be an M-space. A nonempty subset $D \subset X$ is said to be *admissible*, if $D^n \subset M_n$ for all n.

Definition 2.8. Let $(X, \mathbf{M}, \mathbf{k})$ be an M-space, let $D \subset X$ be an admissible subset. We say that a subset S of X is *M-convex with respect to D*, if for each subset $A \in \langle S \cap D\rangle$ and any indexing of $A = \{a_1, \ldots, a_{n+1}\}$, we have that

$$k_{n+1}\big((a_1, \ldots, a_{n+1}), \overline{\Delta}_n\big) \subset S.$$

If $D = X$ we say *M-convex*.

The idea of an MC-space or multiconnected structure was introduced by J.V. Llinares [4].

Notation. Given a topological space X and a subspace $D \subset X$, $\mathbf{F}_D$ will denote the collection of all the functions from $D \times [0,1]$ into D, that is, $\mathbf{F}_D = \{f : D \times [0,1] \to D\}$.

Definition 2.9. An *MC-space* is a pair $(X, \mathbf{P})$ where X is a topological space, and $\mathbf{P} = \{P_A\colon A \in \langle X \rangle\}$ is a collection of finite subsets of $\mathbf{F}_X$ such that given any finite subset $A = \{a_0, \ldots, a_n\}$, P_A is a finite family of functions $P_i^A : X \times [0,1] \to X$ for $i = 0, \ldots, n$ such that

(1) $P_i^A(x, 0) = x$ and $P_i^A(x, 1) = b_i$ for each $x \in X$, and some $b_i \in X$;
(2) The function $G_A : [0,1]^n \to X$ defined by
$$G_A(t_0, \ldots, t_{n-1}) = P_0^A\big(\ldots\big(P_{n-1}^A\big(P_n^A(a_n, 1), t_{n-1}\big), \ldots\big), t_0\big)$$
is continuous for each $(t_0, \ldots, t_{n-1}) \in [0,1]^n$.

Let us now introduce here the concept of an L-space, a somewhat simpler idea than the previous one of MC-space, and we will see later how this new concept is related to G-spaces and M-spaces.

Definition 2.10. An *L-space* is a triple $(X, D, \mathbf{P})$ where X is a topological space, D is a nonempty subspace of X, and $\mathbf{P} = \{P_a\colon a \in X\}$ is a collection of functions $P_a : D \times [0,1] \to D$ such that: $P_a(x, 0) = x$, $P_a(x, 1) = a$; and each P_a is continuous with respect to t.

Definition 2.11. Suppose $(X, D, \mathbf{P})$ is an L-space. Given $A \in \langle D \rangle$, $A = \{a_0, \ldots, a_n\}$, any indexing of A by $\{0, \ldots, n\}$ defines the function $G_A : [0,1]^n \to D$ by
$$G_A(t_0, \ldots, t_{n-1}) = P_{a_0}\big(P_{a_1}\big(\ldots\big(P_{a_{n-2}}\big(P_{a_{n-1}}(a_n, t_{n-1}), t_{n-2}\big), \ldots\big), t_1\big), t_0\big).$$
For $A = \{a\}$, we define $G_{\{a\}} = \{a\}$. We say that a subset $S \subset X$ is an L-convex if for every $A \in \langle S \cap D \rangle$, and every indexing of $A = \{a_0, \ldots, a_m\}$, it follows that $G_A([0,1]^m) \subset S$.

3. Relationship between spaces with abstract convexities

In this section, we shall investigate the relations between the apparently disparate notions introduced in the previous section.

Let us begin with H-spaces and G-spaces.

Theorem 3.1. *Let (X, D, Γ) be an H-space. Then (X, D, Γ) is a G-space. Moreover, every H-convex set is also G-convex and conversely.*

Proof. We know $\Gamma : \langle D \rangle \to X$ is such that $\Gamma(A)$ is contractible for each $A \in \langle D \rangle$. Let $A \in \langle D \rangle$ and let it be indexed thus: $A = \{a_1, \ldots, a_{n+1}\}$. In other words, we choose a bijective function $h : \{1, \ldots, n+1\} \to A$. Then, the composition $\widetilde{\Gamma} = \Gamma \circ h$ clearly satisfies the hypotheses of Theorem 1.3. Thus, there is a continuous function $f : \overline{\Delta}_n \to X$ so that $f([\{e_i\colon i \in J\}]) \subset \widetilde{\Gamma}(J)$ for $J \subset \{1, \ldots, n+1\}$. Observing that for any $B \subset A$, there is $J \subset \{1, \ldots, n+1\}$ so that $B = h(J)$, we simply set $\phi_A = f$ to get
$$\phi_A\big(\big[\{e_i\colon i \in J\}\big]\big) \subset \widetilde{\Gamma}(J) = \Gamma\big(h(J)\big) = \Gamma(B).$$
Finally, it is clear that if S is an H-convex, it is a G-convex and conversely. □

The converse is not true, as the next simple example shows.

Example 1. Let X be a curve with end points x and y, $x \neq y$ as shown in Fig. 1.
Let $D = \{x, y\}$, and $\Gamma : \langle D \rangle \to X$ the map defined as follows:
$$\Gamma\big(\{x\}\big) = \{x\}, \quad \Gamma\big(\{y\}\big) = \{y\}, \quad \text{and} \quad \Gamma\big(\{x, y\}\big) = X.$$

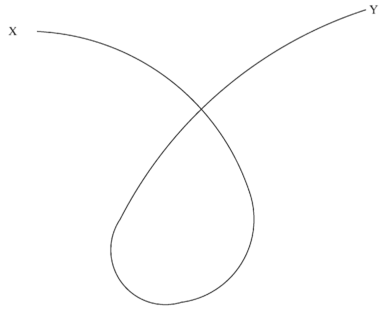

Fig. 1. A triple (X, D, Γ) which is a G-space, but not an H-space.

It is clear that there exists a continuous function f that maps the simplex $[e_1, e_2]$ onto $[\{x, y\}]$, such that $f(e_1) = x$ and $f(e_2) = y$ and also it is clear that there is a continuous function ϕ that maps the segment $[\{x, y\}]$ onto X. Then $\phi \circ f$ is a continuous function from the simplex $[e_1, e_2]$ onto X such that $\phi \circ f(e_1) = x$ and $\phi \circ f(e_2) = y$. Thus, the map Γ satisfies condition (2) of definition of G-space. Therefore (X, D, Γ) is a G-space, but this triple is not an H-space. Indeed $\Gamma(\{x, y\})$ is not contractible.

The next theorem shows the relation between G-spaces and M-spaces.

Theorem 3.2. *Let $(X, \mathbf{M}, \mathbf{k})$ be an M-space. Suppose $D \subset X$ is an admissible subset. Let $\Gamma : \langle D\rangle \to X$ be defined by*

$$\Gamma(A) = \bigcup \big\{k_{n+1}\big((a_0, \dots, a_n), \overline{\Delta}_n\big)\colon\ (a_0, \dots, a_n) \text{ ordered according to an indexing of } A\big\}.$$

Then (X, D, Γ) is a G-space. Moreover, if a set S is an M-convex with respect to D, then it is a G-convex and conversely.

Proof. Suppose that $A \subset B$, we will show that $\Gamma(A) \subset \Gamma(B)$. Indeed, assume $|A| = n + 1$ and $|B| = n + m + 1$. Let $a \in \Gamma(A)$. Then $a = k_{n+1}(s, t)$ for some $s = (y_0, \dots, y_n)$, that is, $A = \{y_0, \dots, y_n\}$. Suppose $B = \{y_0, \dots, y_n, y_{n+1}, \dots, y_{n+m}\}$ and let $q = (y_0, \dots, y_n, y_{n+1}, \dots, y_{n+m})$, $\tilde{t} = (t_0, \dots, t_n, \underbrace{0, \dots, 0}_{m \text{ zeroes}})$, and $t = (t_0, \dots, t_n)$.

The last component of $\tilde{t}$ is zero, then by condition (2) of the definition of M-space, it follows that

$$\begin{aligned} k_{n+m+1}(q, \tilde{t}) &= k_{n+m}(\delta_{n+m+1} q, \delta_{n+m+1} \tilde{t}) \\ &= k_{n+m}\big((y_0, \dots, y_{n+m-1}), (t_0, \dots, t_n, \underbrace{0, \dots, 0}_{m-1 \text{ zeroes}})\big) \quad \text{(now we do the same with the last zero)} \\ &\ \ \vdots \\ &= k_{n+1}(s, t) = a. \end{aligned}$$

Hence $a \in k_{n+m+1}((y_0, \dots, y_{n+m}), \overline{\Delta}_{n+m}) \subset \Gamma(B)$.

Now let us prove that (X, D, Γ) satisfies condition (2) of G-spaces. Let $A \in \langle X\rangle$, say $A = \{a_1, \dots, a_{n+1}\}$, and consider $\bar{x} = (a_1, a_2, \dots, a_{n+1}) \in M_{n+1}$.

Let us define $\phi_A : \overline{\Delta}_n \to \Gamma(A)$ by $\phi_A(t) = k_{n+1}(\bar{x}, t)$. Then by definition of M-space we have that ϕ_A is a continuous function.

On the other hand, suppose $J = \{a_{i_1}, \dots, a_{i_m}\} \subset A$, and let $K = \{i_1, \dots, i_m\} \subset \{1, \dots, n+1\}$. If $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{n+1}) \in [\{e_i\colon i \in K\}]$, then $\tilde{t}_i = 0$ if $i \in \{1, \dots, n+1\} \setminus K$.

Then if $j \in \{1, \dots, n+1\} \setminus K$ it follows that the jth component of $\tilde{t}$ is zero, and by condition (2) of Definition 2.6 we have that

$$\begin{aligned} \phi_A(\tilde{t}) &= k_{n+1}(\bar{x}, \tilde{t}) \\ &= k_n(\delta_j \bar{x}, \delta_j \tilde{t}) \end{aligned}$$

$$= k_n\big((a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_{n+1}), (\tilde{t}_1, \dots, \tilde{t}_{j-1}, \tilde{t}_{j+1}, \dots, \tilde{t}_{n+1})\big)$$
$$\vdots$$
$$= k_m\big((a_{i_1}, \dots, a_{i_m}), (\tilde{t}_{i_1}, \dots, \tilde{t}_{i_m})\big) \in \Gamma(J).$$

So, we have proved that given a set $A \in \langle D\rangle$ with $|A| = n + 1$, there exists a continuous function $\phi_A : \overline{\Delta}_n \to \Gamma(A)$ such that if $J = \{a_{i_1}, \dots, a_{i_m}\} \subset A$, then $\phi_A([\{e_{i_1}, \dots, e_{i_m}\}]) \subset \Gamma(J)$. Thus, (X, D, Γ) is a G-space.

Finally, it is clear that a set S is an M-convex with respect to D if and only if it is a G-convex. □

The following lemma will be used and its proof can be found in [4].

Lemma 3.3. *Let $(X, D, \mathbf{P})$ be an L-space. Let $t_i : \overline{\Delta}_n \to [0, 1]$; $i = 0, \dots, n-1$ functions defined by $t_0(s) = s_0$, $t_1(s) = \frac{s_1}{1-s_0}$, $t_2(s) = \frac{s_2}{1-s_0-s_1}$, $\dots$, $t_{n-1}(s) = \frac{s_{n-1}}{1-s_0-s_1-\dots-s_{n-2}}$, in the case that all the denominators are not zero. If $s_0 + \dots + s_{i-1} = 1$ and $s_{i-1} \neq 0$, we define $t_j = 0$ for $j \geqslant i$ and the other t_j's as above, where $s = (s_0, \dots, s_n) \in \overline{\Delta}_n$. Then, for any finite set $A \in \langle D\rangle$, $A = \{a_0, \dots, a_n\}$ the composition function $\phi_A : \overline{\Delta}_n \to X$, given by*

$$\phi_A(s) = G_{\{a_0,\dots,a_n\}}\big(t_0(s), \dots, t_{n-1}(s)\big)$$

is a continuous function.

The next theorem relates L-spaces and M-spaces.

Theorem 3.4. *Suppose $(X, D, \mathbf{P})$ is an L-space. Let $M_n = D^n$ for $n \geqslant 1$, and define $k_{n+1} : M_{n+1} \times \overline{\Delta}_n \to X$ as follows, if $a = (a_0, \dots, a_n) \in M_{n+1}$ is such that $|\{a_0, \dots, a_n\}| = n + 1$, then*

$$k_{n+1}\big((a_0, \dots, a_n), (s_0, \dots, s_n)\big) = G_{\{a_0,\dots,a_n\}}(t_0, \dots, t_{n-1}),$$

where $t_0 = s_0$, $t_1 = \frac{s_1}{1-s_0}$, $t_2 = \frac{s_2}{1-s_0-s_1}$, $\dots$, $t_{n-1} = \frac{s_{n-1}}{1-s_0-s_1-\dots-s_{n-2}}$, in the case that all the denominators are not zero. If $s_0 + \dots + s_{i-1} = 1$ and $s_{i-1} \neq 0$, we define $t_j = 0$ for $j \geqslant i$ and the other t_j's as above.

If $a = (a_0, \dots, a_n) \in M_{n+1}$ is such that $a_i = a_j$ for some $i < j \leqslant n + 1$ define $k_{n+1}(a, s) = k_n(\delta_i(a), s^*)$, where $s^* = (s_0, \dots, s_{i-1}, s_{i+1}, \dots, s_{j-1}, s_j + s_i, s_{j+1}, \dots, s_n)$.

Then $(X, \mathbf{M}, \mathbf{k})$ is an M-space and D is an admissible subset. Moreover, a subset $S \subset X$ is an L-convex if and only if it is an M-convex with respect to D.

Proof. First, let us assume that $a = (a_0, \dots, a_n) \in M_{n+1}$ is such that $|\{a_0, \dots, a_n\}| = n + 1$. We will show that for $i \leqslant n$ and $s_i = 0$ we have that $k_{n+1}(a, s) = k_n(\delta_i a, \delta_i s)$, where $s \in \overline{\Delta}_n$.

Indeed, suppose $i < n$ and $s_i = 0$, then $t_i = 0$, and we have

$$\begin{aligned}
k_{n+1}(a, s) &= k_{n+1}\big((a_0, \dots, a_n), (s_0, \dots, s_n)\big)\\
&= P_{a_0}\big(P_{a_1}\big(\dots\big(P_{a_{n-2}}\big(P_{a_{n-1}}(a_n, t_{n-1}), t_{n-2}\big), \dots\big), t_1\big), t_0\big)\\
&= P_{a_0}\big(P_{a_1}\big(\dots\big(P_{a_{i-1}}\big(P_{a_{i+1}} \dots P_{a_{n-1}}(a_n, t_{n-1}), \dots t_{i+1}\big), t_{i-1} \dots\big), t_1\big), t_0\big)\\
&\qquad \big(\text{the last equality is because } P_{a_i}(x, 0) = x\big)\\
&= G_{\{a_0,\dots,a_{i-1},a_{i+1},\dots,a_n\}}(t_0, \dots, t_{i-1}, t_{i+1}, \dots, t_{n-1})\\
&= k_n\big((a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_n), (s_0, \dots, s_{i-1}, s_{i+1}, \dots, s_n)\big)\\
&= k_n(\delta_i a, \delta_i s).
\end{aligned}$$

Suppose now that $i = n$ and $s_n = 0$. Recall that $t_{n-1} = \frac{s_{n-1}}{1-s_0-s_1-\dots-s_{n-2}}$, so if $s_n = 0$, then $1 - s_0 - \dots - s_{n-2} = s_{n-1}$ it follows that $t_{n-1} = 1$, and we have

$$\begin{aligned}
k_{n+1}(a, s) &= k_{n+1}\big((a_0, \dots, a_n), (s_0, \dots, s_n)\big)\\
&= P_{a_0}\big(P_{a_1}\big(\dots\big(P_{a_{n-2}}\big(P_{a_{n-1}}(a_n, t_{n-1}), t_{n-2}\big), \dots\big), t_1\big), t_0\big)\\
&= P_{a_0}\big(P_{a_1}\big(\dots\big(P_{a_{n-2}}(a_{n-1}, t_{n-2}), \dots\big), t_1\big), t_0\big)
\end{aligned}$$

$$= k_n\big((a_0, \ldots, a_{n-1}), (s_0, \ldots, s_{n-1})\big)$$
$$= k_n(\delta_i a, \delta_i s).$$

Second, let us assume that $|\{a_0, \ldots, a_n\}| = m < n+1$, and suppose that for $i \leqslant n+1$ we have that $s_i = 0$. Then we have two possibilities.

Either there is $a_j = a_i$, in which case we have that

$$\begin{aligned} k_{n+1}(a, s) &= k_{n+1}\big((a_0, \ldots, a_i, \ldots, a_j, \ldots, a_n), (s_0, \ldots, s_n)\big) \\ &= k_n\big((a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n), (s_0, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)\big) \\ &= k_n\big(\delta_i(a), \delta_i(s)\big). \end{aligned}$$

Or, there is no a_j with $a_i = a_j$. In this case, by using the way in which the functions k's are defined when $|\{a_0, \ldots, a_n\}| < n+1$, we can take the repeated elements out to get

$$k_{n+1}(a, s) = k_m\big((a_{l_1}, \ldots, a_{l_m}), \big(s'_{l_1}, \ldots, s'_{l_m}\big)\big),$$

where $s_i = s'_{l_j} = 0$ for some j. Now, employing the results in the first part we get that

$$k_{n+1}(a, s) = k_{m-1}\big((a_{l_1}, \ldots, a_{l_{j-1}}, a_{l_{j+1}}, \ldots, a_{l_m}), \big(s'_{l_1}, \ldots, s'_{l_{j-1}}, s'_{l_{j+1}}, \ldots, s'_{l_m}\big)\big).$$

Now, using again the definition of the functions k's, we can put back the repeated elements to get $k_{n+1}(a, s) = k_n(\delta_i(a), \delta_i(s))$.

On the other hand, each function $k_n(a, s)$ is continuous with respect to s. In fact, if $|\{a_0, \ldots, a_n\}| = n+1$, $k_{n+1}(a, s) = G_{\{a_0, \ldots, a_n\}}(t_0, \ldots, t_{n-1})$, which is continuous by Lemma 3.3.

If $|\{a_0, \ldots, a_n\}| = m < n+1$, then

$$k_{n+1}\big((a_0, \ldots, a_n), (s_0, \ldots, s_n)\big) = k_{m+1}\big((a_{l_0}, \ldots, a_{l_m}), \big(s'_{l_0}, \ldots, s'_{l_m}\big)\big) = G_{\{a_{l_0}, \ldots, a_{l_m}\}}\big(s'_{l_0}, \ldots, s'_{l_m}\big),$$

where each s' is a continuous function of s, so k_{n+1} is a composition of continuous functions.

Thus, we have proved that $(X, \mathbf{M}, \mathbf{k})$ is an M-space. The fact that $M_n = D^n$ means that D is an admissible subset.

Now suppose S is an L-convex convex subset of $(X, D, \mathbf{P})$, let us show that S is an M-convex subset with respect to D in $(X, \mathbf{M}, \mathbf{k})$.

Let $A = \{a_0, \ldots, a_n\} \in \langle D \cap S\rangle$, and assume that $x \in \Gamma(A)$, then $x = k_{n+1}((a_0, \ldots, a_n), (s_0, \ldots, s_n))$ for some indexing of A and $(s_0, \ldots, s_n) \in \overline{\Delta}_n$, thus,

$$x = G_{\{a_0, \ldots, a_n\}}(t_0, \ldots, t_{n-1}) \quad \text{for some } (t_0, \ldots, t_{n-1}) \in [0, 1]^n.$$

Then $x \in S$ because S is L-convex. Therefore, $\Gamma(A) \subset S$, which means, that S is M-convex with respect to D.

Conversely, assume that S is an M-convex with respect to D in $(X, \mathbf{M}, \mathbf{k})$. We will show that S is an L-convex in $(X, D, \mathbf{P})$.

Indeed, let $x \in G_{\{a_0, \ldots, a_n\}}(t_0, \ldots, t_{n-1})$, then $x \in k_{n+1}((a_0, \ldots, a_n), (s_0, \ldots, s_n))$, where $s_0 = t_0$, $s_1 = t_1(1 - t_0)$, $\ldots$, $s_{n-1} = t_{n-1}(1 - t_{n-2}) \ldots (1 - t_0)$, $s_n = (1 - t_{n-1})(1 - t_{n-2}) \ldots (1 - t_0)$.

We know that $k_{n+1}((a_0, \ldots, a_n), (s_0, \ldots, s_n)) \in S$ because S is an M-convex with respect to M. Thus $G_{\{a_0, \ldots, a_n\}}(t_0, \ldots, t_{n-1}) \in S$ for any indexing $\{a_0, \ldots, a_n\}$ of A and any $(t_0, \ldots, t_{n-1}) \in [0, 1]^n$. That is, S is an L-convex.

The following theorem relates a particular type of H-spaces with L-spaces. □

Theorem 3.5. *Suppose (X, Γ) is an H-space such that for every $A \in \langle X\rangle$, we have $A \subset \Gamma(A)$. For $a \in X$, define $P_A : X \times [0, 1] \to X$ as follows*

$$P_a(x, t) = \begin{cases} H_{\{a,x\}}(x, t) & \text{if } x \notin \Gamma(\{a\}), \\ H_{\{a\}}(x, t) & \text{if } x \in \Gamma(\{a\}), \end{cases}$$

where $H_{\{a,x\}}$ is a homotopy between the identity and the constant a relative to $\Gamma(\{a, x\})$ and $H_{\{a\}}$ is a homotopy between the identity and the constant a relative to $\Gamma(\{a\})$. Then $(X, \mathbf{P})$, where $\mathbf{P} = \{P_a\colon a \in X\}$ is an L-space.

Proof. Note that $P_a(x,0) = H_{\{a,x\}}(x,0)$ or $H_{\{a\}}(x,0)$, and either case, $P_a(x,0) = x$ for all $x \in X$. Similarly, we know that $P_a(a,1) = a$.

Next, consider $P_a(\tilde{x},t)$ for some fixed $\tilde{x} \in X$. Again, $P_a(\tilde{x},t) = H_{\{a,x\}}(\tilde{x},0)$ or $H_{\{a\}}(\tilde{x},0)$, and in either case, $P_a(\tilde{x},t)$ is a continuous function of t. □

Theorem 3.6. *Let (X,Γ) be an H-space such that for every $A \in \langle X\rangle$ we have $A \subset \Gamma(A)$, and $(X,\mathbf{P})$ the corresponding L-space given by Proposition 3.5. Suppose further that Γ is such that $z \in \Gamma(A)$ implies $\Gamma(A) = \Gamma(A \cup \{z\})$.Then if $S \subset X$ is an H-convex, it must be L-convex.*

Proof. Suppose $S \subset X$ is an H-convex. Let $A = \{a_0,\dots,a_n\} \in \langle S\rangle$ and for $\tilde{t} = (t_0,\dots,t_n) \in [0,1]^n$ define $k_n, k_{n-1},\dots,k_0$ as follows:

$$\begin{aligned}
&k_n = a_n,\\
&k_{n-1} = P_{a_{n-1}}(k_n, t_{n-1}),\\
&\vdots\\
&k_{n-j} = P_{a_{n-j}}(k_{n-j+1}, t_{n-j}),\\
&\vdots\\
&k_1 = P_{a_1}(k_2, t_1),\\
&k_0 = P_{a_0}(k_1, t_0).
\end{aligned}$$

We shall prove that for $j = 0,1,\dots,n$, we have

$$k_{n-j} \in \Gamma(A).$$

First, for $j = 0$ we have $k_n = a_n \in A \subset \Gamma(A)$. Suppose $k_{n-j+1} \in \Gamma(A)$. Then

$$k_{n-j} = P_{a_{n-j}}(k_{n-j+1}, t_{n-j}) \in \Gamma\big(\{a_{n-j}, k_{n-j+1}\}\big).$$

But $\{a_{n-j}, k_{n-j+1}\} \subset A \cup \{k_{n-j+1}\}$. Thus,

$$\Gamma\big(\{a_{n-j}, k_{n-j+1}\big) \subset \Gamma\big(A \cup \{k_{n-j+1}\}\big) = \Gamma(A)$$

and we have $k_{n-j} \in \Gamma(A)$.

Now simply observe that $k_0 = G_A(t_0,\dots,t_n)$ to conclude that $G_A([0,1]^n) \subset \Gamma(A)$. Hence, if S is an H-convex, it must be L-convex. □

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